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A Similarity Law for Stressing Rapidly Heated Thin-Walled Cylinders[1]

H. S. TSIEN[2] and C. M. CHENG[3]

Daniel and Florence Guggenheim Jet Propulsion Center
California Institute of Technology, Pasadena, Calif.

When a thin cylindrical shell of uniform thickness is very rapidly heated by hot high-pressure gas flowing inside the shell, the temperature of material decreases steeply from a high temperature at the inside surface to ambient temperatures at the outside surface. Young's modulus of material thus varies. The purpose of the present paper is to reduce the problem of stress analysis of such a cylinder to an equivalent problem in conventional cylindrical shell without temperature gradient in the wall. The equivalence concept is expressed as a series of relations between the quantities for the hot cylinder and the quantities for the cold cylinder. These relations give the similarity law whereby strains for the hot cylinder can be simply deduced from measured strains on the cold cylinder and thus greatly simplify the problem of experimental stress analysis.

THE cylinder of a solid propellant rocket is subjected to very rapid heating during its short duration of operation. The temperature distribution across the thin cylindrical wall, although approximately the same in every section, is not linear. This condition is most severe at the end of the combustion of the propellant grain. From the point of view of a materials engineer, this case is distinguished from others by the time rate of heating, which is so large as to not allow sufficient time for appreciable change in the structure of the material. The strength of the wall material under this operating condition is quite different from that under slow heating. This fact is clearly and conclusively shown by R. L. Noland in a recent paper (1).[4] From the point of view of a stress analyst, the rational design of a solid propellant rocket cylinder is thus complicated by the very large thermal stress and variable Young's modulus of the material across the wall as a result of the large temperature gradient. Furthermore, experimental stress determination under actual firing tests is rather difficult due to the short test time available and the high temperature.

It is believed that for these reasons the only case which has been analyzed by reliable rational method is the case of rocket cylinder under uniform internal pressure. The bending stresses due to canted nozzles, the stresses due to end enclosures, mounting lugs, etc., are only estimated by very rough methods. The purpose of this paper is to improve this situation by suggesting an approach which will reduce the general stress problem of hot cylinder to a problem of an equivalent cold cylinder. The equivalent problem can then be solved either analytically by the conventional method or directly by experimental stress determination. In either choice, the problem is believed to be greatly simplified. This law of equivalence between hot cylinder and cold cylinder can be called the similarity law.

Stresses and Strains of a Thin-Walled Cylinder

The fact that the thickness of the cylinder is small in comparison to its radius and length allows a great simplification in the strain analysis. To wit, the deformation of every point of the cylinder can be described sufficiently accurately by the displacements of the points on a single surface within the wall of the cylinder. This surface is the so-called median surface. The position of the median surface is so determined that a bending of the median surface will not induce net extensional forces in the plane of the median surface, across the thickness of the wall. When Young's modulus is a constant, as is the case for a cold cylinder, the median surface lies midway between the outer and the inner boundary surfaces of the cylinder. When Young's modulus is not a constant but decreases with increase in temperature, the median surface is displaced toward the cold side, as will be seen presently.

Let x, θ, z be the coordinate system with origin on the cylindrical median surface such that x points in the axial direction of the cylinder, θ is in the circumferential direction, measured on the median surface, and z is normal to the median surface, pointing toward the axis of the cylinder. Let U, V, and W be displacements of a point (x, θ) on the median surface in the directions x, θ, and z, respectively. They are thus functions of x and θ; but not of z. Then the above-mentioned fundamental simplification of thin shells can be stated as follows: If the direct strains in the x and θ directions are e_x and e_θ, and the shear strain $\gamma_{x\theta}$, then

$$\left.\begin{aligned} e_x &= \frac{\partial U}{\partial x} - z\frac{\partial^2 W}{\partial x^2} \\ e_\theta &= \frac{1}{R}\frac{\partial V}{\partial \theta} - \frac{W}{R} - \frac{z}{R^2}\left(\frac{\partial V}{\partial \theta} + \frac{\partial^2 W}{\partial \theta^2}\right) \\ \gamma_{x\theta} &= \frac{\partial V}{\partial x} + \frac{1}{R}\frac{\partial U}{\partial \theta} - 2\frac{z}{R}\left(\frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial V}{\partial x}\right) \end{aligned}\right\} \quad \ldots [1]$$

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[2] Robert H. Goddard Professor of Jet Propulsion.

[3] Graduate Assistant in Mechanical Engineering.

[4] Numbers in parentheses refer to Bibliography on page 162.

where R is the radius of the median cylindrical surface, or the "radius of the cylinder." This result is sometimes described as the Kirchhoff bending assumption: Plane normal to the median surface before bending remains so after bending.

The significant stresses in a thin shell are the direct stresses σ_x and σ_θ in x and θ, and the shear stress $\tau_{x\theta}$. All other stresses are small in comparison to these three. Now let T be the temperature of wall above a reference temperature, say the room temperature. T is assumed to be only a function of thickness ordinate z, but not of x and θ. Thus the heating of the cylinder is assumed to be uniform over the entire surface of the cylinder. This is generally very closely approximated in reality. If α is the coefficient of thermal expansion, the thermal expansion strain is then αT. By Hooke's law one has

$$\left.\begin{aligned} e_x &= \frac{1}{E}(\sigma_x - \nu\sigma_\theta) + \alpha T \\ e_\theta &= \frac{1}{E}(\sigma_\theta - \nu\sigma_x) + \alpha T \\ \gamma_{x\theta} &= \frac{2(1+\nu)}{E}\tau_{x\theta} \end{aligned}\right\} \quad [2]$$

where E is Young's modulus, and ν is Poisson's ratio. E is, of course, a function of temperature or a function of z. ν, however, will be assumed to be constant for lack of definite information. By solving for the stresses, one obtains from Equation [2],

$$\left.\begin{aligned} \sigma_x &= \frac{E(z)}{1-\nu^2}\{(e_x + \nu e_\theta) - (1+\nu)\alpha T(z)\} \\ \sigma_\theta &= \frac{E(z)}{1-\nu^2}\{(e_\theta + \nu e_x) - (1+\nu)\alpha T(z)\} \\ \tau_{x\theta} &= \frac{E(z)}{2(1+\nu)}\gamma_{x\theta} \end{aligned}\right\} \quad [3]$$

For thin-walled cylinders, the equilibrium equations are expressed in terms of "sectional averages" of the stresses given in Equation [3]. That is, one speaks of the normal forces N_x and N_θ, the shearing force $N_{x\theta}$, the bending moments M_x, M_θ, and the twisting moment $M_{x\theta}$. They are related to σ_x, σ_θ, and $\tau_{x\theta}$ by the following equations

$$N_x = \int \sigma_x dz \qquad N_\theta = \int \sigma_\theta dz \qquad N_{x\theta} = \int \tau_{x\theta} dz \quad [4]$$

$$M_x = -\int \sigma_x z dz \qquad M_\theta = -\int \sigma_\theta z dz \qquad M_{x\theta} = -\int \tau_{x\theta} z dz \quad [5]$$

The integrals in the above equations all extend across the thickness of the wall. By substituting Equations [1] and [3] into Equations [4] and [5], one has for example

$$N_x = D_0\left(\frac{\partial U}{\partial x} + \frac{\nu}{R}\frac{\partial V}{\partial \theta} - \nu\frac{W}{R}\right) - D_1\left(\frac{\partial^2 W}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial V}{\partial \theta} + \frac{\nu}{R^2}\frac{\partial^2 W}{\partial \theta^2}\right) - N_T$$

and

$$-M_x = D_1\left(\frac{\partial U}{\partial x} + \frac{\nu}{R}\frac{\partial V}{\partial \theta} - \nu\frac{W}{R}\right) - D_2\left(\frac{\partial^2 W}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial V}{\partial \theta} + \frac{\nu}{R^2}\frac{\partial^2 W}{\partial \theta^2}\right) - M_T$$

where

$$\left.\begin{aligned} D_0 &= \frac{1}{1-\nu^2}\int E(z)dz \\ D_1 &= \frac{1}{1-\nu^2}\int E(z)zdz \end{aligned}\right\} \quad [6]$$

$$D_2 = \frac{1}{1-\nu^2}\int E(z)z^2dz \quad [7]$$

and

$$N_T = \frac{\alpha}{1-\nu}\int E(z)T(z)dz \quad [8]$$

$$M_T = \frac{\alpha}{1-\nu}\int E(z)T(z)zdz \quad [9]$$

The integrals again extend across the thickness of the wall. It is evident from the above expressions for N_x and M_x that considerable simplification can be achieved by choosing the median surface in such a way that

$$D_1 = \frac{1}{1-\nu^2}\int E(z)zdz = 0 \quad [10]$$

This is actually the condition to locate the median surface. Since Young's modulus E decreases with increase in temperature, it is seen from Equation [10] that the median surface is nearer to the cold boundary surface than to the hot boundary surface. For a rocket cylinder, hot inside but cold outside, the median surface is near to the outside surface. With this choice of the median surface, the forces and the moments are related to the displacements by the following simpler equations:

$$\left.\begin{aligned} N_x &= D_0\left(\frac{\partial U}{\partial x} + \frac{\nu}{R}\frac{\partial V}{\partial \theta} - \nu\frac{W}{R}\right) - N_T, \\ N_\theta &= D_0\left(\frac{1}{R}\frac{\partial V}{\partial \theta} - \frac{W}{R} + \nu\frac{\partial U}{\partial x}\right) - N_T \\ N_{x\theta} &= \frac{1-\nu}{2}D_0\left(\frac{\partial V}{\partial x} + \frac{1}{R}\frac{\partial U}{\partial \theta}\right) \end{aligned}\right\} \quad [11]$$

$$\left.\begin{aligned} M_x &= D_2\left(\frac{\partial^2 W}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\nu}{R^2}\frac{\partial V}{\partial \theta}\right) + M_T, \\ M_\theta &= D_2\left(\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial V}{\partial \theta} + \nu\frac{\partial^2 W}{\partial x^2}\right) + M_T \\ M_{x\theta} &= (1-\nu)D_2\left(\frac{1}{R^2}\frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial V}{\partial x}\right) \end{aligned}\right\} \quad [12]$$

It is worthwhile to point out two facts: Firstly, the choice of reference temperature is quite arbitrary. Changing the reference temperature will change the value of N_T, Equation [8]. But a corresponding adjustment in the normal strain will leave the normal forces N_x, N_θ given by Equation [11] unchanged. Therefore the physical problem is quite independent of the choice of reference temperature. M_T is independent of the reference temperature due to Equation [10]. Secondly, the present system of equations reduces to those of the conventional thin shell theory if temperature gradient is absent, or if Young's modulus is independent of the temperature of material. In that case the median surface is midway between the boundary surfaces, and D_2 is simply the usual flexural rigidity of the shell.

Nondimensional Quantities and Equations of Equilibrium

It is useful to introduce nondimensional quantities by taking R as the reference length. Thus

$$\xi = x/R \quad [13]$$

$$u = U/R, \quad v = V/R, \quad w = W/R \quad [14]$$

$$n_\xi = N_x/D_0, \quad n_\theta = N_\theta/D_0, \quad n_{\xi\theta} = N_{x\theta}/D_0, \quad n_T = N_T/D_0 \quad [15]$$

and

$$m_\xi = M_xR/D_2, \quad m_\theta = M_\theta R/D_2, \quad m_{\xi\theta} = M_{x\theta}R/D_2, \quad M_T = M_TR/D_2 \quad [16]$$

Therefore Equations [11] and [12] become now

$$\left.\begin{aligned} n_\xi &= \frac{\partial u}{\partial \xi} + \nu\frac{\partial v}{\partial \theta} - \nu w - n_T, \qquad n_\theta = \frac{\partial v}{\partial \theta} - w + \nu\frac{\partial u}{\partial \xi} - n_T \\ n_{\xi\theta} &= \frac{1-\nu}{2}\left(\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \theta}\right) \end{aligned}\right\} \quad [17]$$

and

$$\left.\begin{aligned} m_\xi &= \frac{\partial^2 w}{\partial \xi^2} + \nu\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial v}{\partial \theta} + m_T, \\ m_\theta &= \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} + \nu\frac{\partial^2 w}{\partial \xi^2} + m_T \\ m_{\xi\theta} &= (1-\nu)\left(\frac{\partial^2 w}{\partial \xi\partial\theta} + \frac{\partial v}{\partial \xi}\right) \end{aligned}\right\} \quad [18]$$

The equations of equilibrium in terms of forces and moments are here exactly the same as the conventional theory (2). The only innovation is to write them in nondimensional form. For this purpose, one has to define the nondimensional quantities q_ξ, q_θ, and p of the dimensional sectional shearing forces Q_x, Q_θ and normal pressure loading P, against z-direction as follows:

$$q_\xi = Q_x/D_0 \qquad q_\theta = Q_\theta/D_0 \qquad p = PR/D_0 \quad [19]$$

Then the equilibrium equations of forces are

$$\left.\begin{aligned} &\frac{\partial n_\xi}{\partial \xi} + \frac{\partial n_{\xi\theta}}{\partial \theta} - q_\xi\frac{\partial^2 w}{\partial \xi^2} - q_\theta\left(\frac{\partial v}{\partial \xi} + \frac{\partial^2 w}{\partial \xi\partial\theta}\right) - n_{\xi\theta}\frac{\partial^2 v}{\partial \xi^2} - n_\theta\left(\frac{\partial^2 v}{\partial \xi\partial\theta} - \frac{\partial w}{\partial \xi}\right) = 0 \\ &\frac{\partial n_{\xi\theta}}{\partial \xi} + \frac{\partial n_\theta}{\partial \theta} - q_\xi\left(\frac{\partial v}{\partial \xi} + \frac{\partial^2 w}{\partial \xi\partial\theta}\right) - q_\theta\left(1 + \frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) + n_\xi\frac{\partial^2 v}{\partial \xi^2} + n_{\xi\theta}\left(\frac{\partial^2 v}{\partial \xi\partial\theta} - \frac{\partial w}{\partial \xi}\right) = 0 \\ &\frac{\partial q_\xi}{\partial \xi} + \frac{\partial q_\theta}{\partial \theta} + 2n_{\xi\theta}\left(\frac{\partial v}{\partial \xi} + \frac{\partial^2 w}{\partial \xi\partial\theta}\right) + n_\xi\frac{\partial^2 w}{\partial \xi^2} + n_\theta\left(1 + \frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) = p \end{aligned}\right\} \quad [20]$$

The equations for the equilibrium of moments are

$$\left.\begin{aligned} &\frac{\partial m_{\xi\theta}}{\partial \xi} + \frac{\partial m_\theta}{\partial \theta} + m_\xi\frac{\partial^2 v}{\partial \xi^2} + m_{\xi\theta}\left(\frac{\partial^2 v}{\partial \xi\partial\theta} - \frac{\partial w}{\partial \xi}\right) + \beta q_\theta = 0 \\ &-\frac{\partial m_\xi}{\partial \xi} - \frac{\partial m_{\xi\theta}}{\partial \theta} + m_{\xi\theta}\frac{\partial^2 v}{\partial \xi^2} + m_\theta\left(\frac{\partial^2 v}{\partial \xi\partial\theta} - \frac{\partial w}{\partial \xi}\right) - \beta q_\xi = 0 \end{aligned}\right\} \quad [21]$$

where

$$\beta = R^2D_0/D_2 \quad [22]$$

β is thus a quantity of the order of $(R/b)^2$, b the thickness of the wall. There are eleven individual equations in the system of Equations [17], [18], [20], and [21]. With specified loading p, there are also eleven unknowns, u, v, w, n_ξ, n_θ, $n_{\xi\theta}$, m_ξ, m_θ, $m_{\xi\theta}$, q_ξ and q_θ. The system of equations is thus complete.

Infinite Cylinder Under Uniform Internal Pressure

The simplest special case in the present general problem is the case of very long cylinder under uniform internal pressure. If the rocket cylinder is long in comparison with its diameter, the actual stress system during operation is approximated by this idealized simple case. In this problem of infinitely long uniformly loaded cylinder, the forces n_ξ, n_θ, and moments m_ξ, m_θ are constants independent of ξ and θ. The shearing force $n_{\xi\theta}$ and the twisting moment $m_{\xi\theta}$ vanish. u is proportional to ξ or $\frac{\partial u}{\partial \xi}$ is a constant, say k_1. v vanishes. w is a constant, and is negative in the present coordinate system, say $-k_2$. Then Equations [17], [18], and [20] give

$$\left.\begin{aligned} n_\xi^0 &= k_1 + \nu k_2 - n_T \\ n_\theta^0 &= \nu k_1 + k_2 - n_T \\ m_\xi^0 &= m_\theta^0 = m_T \end{aligned}\right\} \quad [23]$$

where the superscript 0 denote the quantities in the present simple stress system. When the temperature distribution and the material properties are specified, Equation [23] gives the strains k_1 and k_2 in terms of the internal pressure p, and the axial load n_ξ. If the axial load is produced by the same internal pressure, then it can easily be shown that

$$n_\xi^0 = p^0/2 \quad [24]$$

It is of interest to note that the bending moment m_ξ and m_θ are equal to m_T and are independent of the conditions of loading.

By solving Equation [23] for k_1 and k_2, one has

$$\left.\begin{aligned} k_1 &= \frac{1}{1-\nu^2}(n_\xi^0 - \nu p^0) + \frac{1}{1+\nu}n_T \\ k_2 &= \frac{1}{1-\nu^2}(p^0 - \nu n_\xi^0) + \frac{1}{1+\nu}n_T \end{aligned}\right\} \quad [25]$$

If the design condition is the maximum strain of the material, then Equation [25] gives the criterion directly from the pressure and temperature loading.

Linearized Theory for General Secondary Loading

As stated in the previous section, the actual stress system in a rocket chamber is approximately that of infinitely long cylinder under uniform internal pressure. This stress system can be called the primary stresses. The deviations from the primary stress system are results of bending due to canted nozzle, to end enclosures, to mounting lugs, etc. These additional stresses or the secondary stresses are, however, only a fraction of the primary stresses. Therefore it is justified to consider the second order terms of additional stresses and deformations as small in comparison to the first order terms, and thus negligible. In other words,

$$
\left.\begin{array}{llll}
u = k_1\xi + u' & v = v' & w = -k_2 + w' \\
n_\xi = n_\xi^0 + n_\xi' & n_\theta = p^0 + n_\theta' & n_{\xi\theta} = n_{\xi\theta}' \\
m_\xi = m_T + m_\xi' & m_\theta = m_T + m_\theta' & m_{\xi\theta} = m_{\xi\theta}' \\
\text{and} & & \\
q_\xi = q_\xi' & q_\theta = q_\theta' & p = p^0 + p'
\end{array}\right\} \quad [26]
$$

where k_1 and k_2 are given by Equation [25]. The primed quantities are then the secondary deformations and the secondary stresses, they are considered to be small in comparison to the primary deformations and stresses. From Equations [17] and [18], one has the following relations between the deformations and the stresses,

$$
\left.\begin{array}{l}
n_\xi' = \dfrac{\partial u'}{\partial \xi} - \nu w' + \nu \dfrac{\partial v'}{\partial \theta}, \qquad n_\theta' = \nu \dfrac{\partial u'}{\partial \xi} - w' + \dfrac{\partial v'}{\partial \xi}, \\
n_{\xi\theta}' = \dfrac{1-\nu}{2}\left(\dfrac{\partial v'}{\partial \xi} + \dfrac{\partial u'}{\partial \theta}\right) \\
\text{and} \\
m_\xi' = \dfrac{\partial^2 w'}{\partial \xi^2} + \nu \dfrac{\partial^2 w'}{\partial \theta^2} + \nu \dfrac{\partial v'}{\partial \theta}, \\
m_\theta' = \nu \dfrac{\partial^2 w'}{\partial \xi^2} + \dfrac{\partial^2 w'}{\partial \theta^2} + \dfrac{\partial v'}{\partial \theta}, \\
m_{\xi\theta}' = (1-\nu)\left(\dfrac{\partial^2 w'}{\partial \xi \partial \theta} + \dfrac{\partial v'}{\partial \xi}\right)
\end{array}\right\} \quad [27]
$$

By substituting Equation [26] into the equilibrium equations, Equations [20], and dropping out the second order terms of the primed quantities, a system of linearized equations is obtained. This system can be further reduced by substituting the q_ξ' and q_θ' obtained from the last two equations of bending moment equilibrium into the third equation. The final result is the following system of three equations: The first is the equilibrium of forces in the axial direction; the second is the equilibrium of forces in the circumferential direction; and the third is the equilibrium of forces in the radial direction:

$$
\left.\begin{array}{l}
\dfrac{\partial^2 u'}{\partial \xi^2} + \dfrac{1-\nu}{2}\dfrac{\partial^2 u'}{\partial \theta^2} + \dfrac{1+\nu}{2}\dfrac{\partial^2 v'}{\partial \xi \partial \theta} - \nu \dfrac{\partial w'}{\partial \xi} = 0 \\
\dfrac{\partial^2 v'}{\partial \theta^2} + \dfrac{1-\nu}{2}\dfrac{\partial^2 v'}{\partial \xi^2} + \dfrac{1+\nu}{2}\dfrac{\partial^2 u'}{\partial \xi \partial \theta} - \dfrac{\partial w'}{\partial \xi} = 0 \\
\text{and} \\
\dfrac{\partial^4 w'}{\partial \xi^4} + 2\dfrac{\partial^4 w'}{\partial \xi^2 \partial \theta^2} + \dfrac{\partial^2 w'}{\partial \theta^4} - \beta\left(\nu \dfrac{\partial w'}{\partial \xi} - w' + \dfrac{\partial v'}{\partial \theta}\right) = \\
\qquad -\beta p' + \beta\left(n_\xi^0 - \dfrac{m_T}{\beta}\right)\dfrac{\partial^2 w'}{\partial \xi^2} + \beta p^0 \dfrac{\partial^2 w'}{\partial \theta^2}
\end{array}\right\} \quad [28]
$$

These equations have been simplified on the basis that β is a large quantity of the order of the square of the radius-thickness ratio.

In the latter equation, p' is the secondary load imposed on the cylinder expressed as a distributed pressure over the surface of the cylinder, directed radially outward. If the load is a concentrated load, then it has to be expanded into a product of Fourier series and Fourier intergral as done by S. W. Yuan (3) in his treatment of concentrated load on a cold cylindrical shell. Other types of loads can be similarly expanded. Then Equation [28] is a system of three equations for three unknowns u', v', and w'. The forces and moments are related to these displacements by Equation [27].

The problem of general secondary loading as expressed by Equations [27] and [28] is very similar to the problem of general loading on a cold cylindrical shell, and can thus be treated by the known methods developed for this conventional problem. In fact, the only difference between the hot cylinder and the cold cylinder is the appearance of the term m_T in Equation [28]. However, even this difference is trivial: The reason is the very large magnitude of β as shown by Equations [6], [7], and [22]. In fact, if, as is generally the case, N_T is of the same order of magnitude as N_x^0, then the above-cited equations show that the ratio of m_T/β and n_ξ^0 is at most of the order of b/R, where b is the thickness of the shell. Since the shell is considered to be thin, or $b/R \ll 1$, the terms involving m_T in Equation [28] can be dropped without impairing the accuracy of the present theory. When this is done, then in the system of equations given by Equations [27] and [28], the effects of thermal stresses and variable Young's modulus are not explicit. As far as the nondimensional equations are concerned, the problem of hot cylinder is identical to the problem of cold cylinder, and the basic equations are now essentially the same as that adopted by L. H. Donnell for his study of the stability of thin cylindrical shells (4). This is the basis of the similarity law discussed in the following section.

Similarity Law for General Loading

If the problem of secondary loading is to be solved analytically, the results of the previous section show that it can be reduced to an equivalent problem of cold cylinder and solved accordingly. However, a more useful application of the equivalence concept lies in the possibility of experimentally determining the stress and the strain on the equivalent cold cylinder and then using the similarity law to determine the stress and the strain in the hot cylinder. There are mainly two advantages of this semi-experimental approach: (*a*) The experiments on a cold cylinder can be done more easily and much more accurately than possible on a hot cylinder. The test period can be as long as desired and not limited to the short burning time of the rocket. (*b*) The stresses induced by mounting lugs, etc., are very difficult to approximate by simple load systems amenable to theoretical calculations. For instance, the loads from a mounting lug are not really a concentrated force and a concentrated moment. To take them as a concentrated force and a concentrated moment would grossly overestimate the actual stress. Such difficulties disappear if the loading is done experimentally.

With such experimental stress determination in mind, it will be convenient to have the hot cylinder and the equivalent cold cylinder of same general sizes. Thus the radius R and the length L will be the same for both cylinders. In order that the nondimensional differential equations, Equation [28], be the same for the hot cylinder as for the cold cylinder, the parameters in these differential equations must be the same. That is, if a quantity of the cold cylinder is denoted by a bar over the quantity

$$\frac{1}{R^2}\beta = \frac{D_0}{D_2} = \frac{\bar{D}_0}{\bar{D}_2} \qquad [29]$$

$$n_\xi^0 = \frac{N_x^0}{D_0} = \frac{\bar{N}_x^0}{\bar{D}_0}, \qquad p^0 = \frac{P^0}{D_0} = \frac{\bar{P}^0}{\bar{D}_0} \qquad [30]$$

The condition of Equation [29] can be satisfied by making the thickness $\bar{b}$ of the cold cylinder smaller than the thickness b of the hot cylinder. This is of course to be expected, since Young's modulus of the hot material is smaller and hence material is "softer" than the cold material. When the thickness $\bar{b}$ is determined from b by using Equation [29], $\bar{D}_0$ can be computed. Then Equation [30] gives the internal pressure $\bar{P}^0$ and the axial load $\bar{N}_x^0$ from the specified P^0 and N_x^0 for the hot cylinder. These steps then fix the geometry of the cold cylinder and the primary system of loads.

For the additional secondary loads, the fact that Equations [27] and [28] are linear equations can be utilized to introduce an added freedom in specifying the loads. Linear relations are not altered by multiplying the variable by a constant. Therefore for additional loads and additional displacements, the nondimensional quantities for the cold cylinder and the nondimensional quantities for the hot cylinder need not be identical, but differ by a factor ϵ. Thus

$$(\bar{u}', \bar{v}', \bar{w}') = \epsilon(u', v', w')$$

and

$$(\bar{n}_\xi', \bar{n}_\theta', \bar{n}_{\xi\theta}'; \bar{m}_\xi', \bar{m}_\theta', \bar{m}_{\xi\theta}') = \epsilon(n_\xi', n_\theta', n_{\xi\theta}'; m_\xi', m_{\theta\xi}', m_\theta') \qquad [31]$$

Then

$$\bar{p}' = \epsilon p' \qquad [32]$$

But the nondimensional pressure loading p is related to the actual pressure loading by Equation [19]. Therefore the secondary pressure loadings $\bar{P}'$ for the cold cylinder and the secondary pressure loading P' for the hot cylinder are related through

$$\bar{P}' = \left(\frac{\bar{D}_0}{D_0}\epsilon\right) P' \qquad [33]$$

The ratio of the pressure load is then $\bar{D}_0\epsilon/D_0$. Since the radius R and the length L of both cylinders are the same, other types of loads such as concentrated force, or moment for the cold cylinder and for the hot cylinder must also bear the same ratio. Needless to say, the loads for the cold cylinder must be applied at corresponding points for loads in the hot cylinder.

The additional forces N_x', $N_{x\theta}'$, the additional shear Q_x', and the additional moments N_x', $M_{x\theta}'$ at the ends of the cylinder are controlled by Equations [15], [16], and [19]. It is easily seen that because of Equation [29], the ratio of these quantities for the cold cylinder and the hot cylinder is again $\bar{D}_0\epsilon/D_0$.

Therefore, knowing the load system on the hot cylinder, one can find the corresponding load system for the cold cylinder. The factor ϵ for the secondary loads can be chosen at convenience of the experimenter. For instance, ϵ might be so chosen as to make the ratio $\bar{D}_0\epsilon/D_0$ equal to unity. Then the secondary load system for the cold cylinder is exactly the same as the hot cylinder. When the proper load for the cold cylinder is selected and the corresponding strain on the cold cylinder determined by strain gages, the inverse equivalence problem is then to find the strain in the hot cylinder from the test data on cold cylinder.

Take for instance, the axial strain $e_x(z)$. For the cold cylinder, according to Equations [1] and [25],

$$\bar{e}_x(z) = k_1 - \frac{1}{1+\nu}n_T + \left(\frac{\partial \bar{u}'}{\partial \xi} - \frac{\bar{z}}{R}\frac{\partial^2 \bar{w}'}{\partial \xi^2}\right) \qquad [34]$$

where $\bar{z}$ is the value of z measured from the median surface of the cold cylinder, midway between the boundary surfaces. Now let $\bar{e}_x$ be the average of the measured axial strains on the outer surface and on the inner surface of the cold cylinder, and let $\Delta\bar{e}_x$ be the difference of the measured axial strains on the outer surface and on the inner surface of the cold cylinder. Then from Equation [34]

$$\bar{e}_x = k_1 - \frac{1}{1+\nu}n_T + \frac{\partial \bar{u}'}{\partial \xi} = k_1 - \frac{1}{1+\nu}n_T + \epsilon\frac{\partial u'}{\partial \xi} \qquad [35]$$

and

$$\Delta\bar{e}_x = \frac{\bar{b}}{R}\frac{\partial^2 \bar{w}'}{\partial \xi^2} = \frac{\bar{b}}{R}\epsilon\frac{\partial^2 w'}{\partial \xi^2}$$

For the hot cylinder, the axial strain is given by

$$e_x(z) = k_1 + \frac{\partial u'}{\partial \xi} - \frac{z}{R}\frac{\partial^2 w'}{\partial \xi^2} \qquad [36]$$

By eliminating $\dfrac{\partial u'}{\partial \xi}$ and $\dfrac{\partial^2 w'}{\partial \xi^2}$ from Equations [35] and [36], one has

$$e_x(z) = \left(1 - \frac{1}{\epsilon}\right)k_1 + \frac{1}{(1+\nu)\epsilon}n_T + \frac{1}{\epsilon}\left(\bar{e}_x - \frac{z}{\bar{b}}\Delta\bar{e}_x\right) \qquad [37]$$

The value of z is measured radially inward from the median surface of the hot cylinder and thus is larger in magnitude for the inside surface than for the outside surface.

Similarly,

$$e_\theta(z) = \left(1 - \frac{1}{\epsilon}\right)k_2 + \frac{1}{(1+\nu)\epsilon}n_T + \frac{1}{\epsilon}\left(\bar{e}_\theta - \frac{z}{\bar{b}}\Delta\bar{e}_\theta\right) \qquad [38]$$

and

$$\gamma_{x\theta}(z) = \frac{1}{\epsilon}\left(\bar{\gamma}_{x\theta} - \frac{z}{\bar{b}}\Delta\bar{\gamma}_{x\theta}\right) \qquad [39]$$

where $\bar{e}_\theta$ is the average of the measured circumferential strain on the outer surface and the inner surface of the cold cylinder, $\Delta\bar{e}_\theta$ is the difference of these strains; $\bar{\gamma}_{x\theta}$ and $\Delta\bar{\gamma}_{x\theta}$ are the corresponding quantities for the shearing strain. In Equations [37] and [38], k_1 and k_2, and n_T are the primary strains computed from Equations [25] and [8]. Therefore these equations allow the calculation of the strains in the hot cylinder from test results from cold cylinder, and thus complete the desired similarity law.

For a stress analyst, the next step is perhaps the calculation of the principle strains at each value of z in the shell, and examine whether the larger of these principal strains exceeds the design limit of the material at the temperature prevailing at that point.

Example of Dimensioning the Equivalent Cold Cylinder

As an example of the procedure outline in the previous section, the data given by Noland (1) will be used to find the equivalent cold cylinder. The temperature distribution in the wall is taken from Fig. 2 of that paper and is reproduced as Fig. 1 here. The material is assumed to be 19-9 DL, and the variation of Young's modulus with temperature is plotted in Fig. 2, again

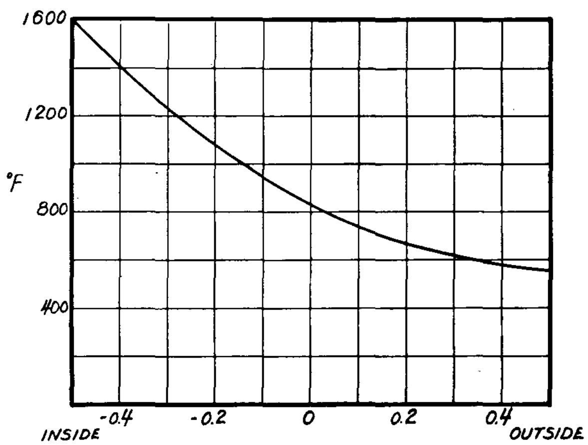

FIG. 1 TEMPERATURE DISTRIBUTION IN THE WALL

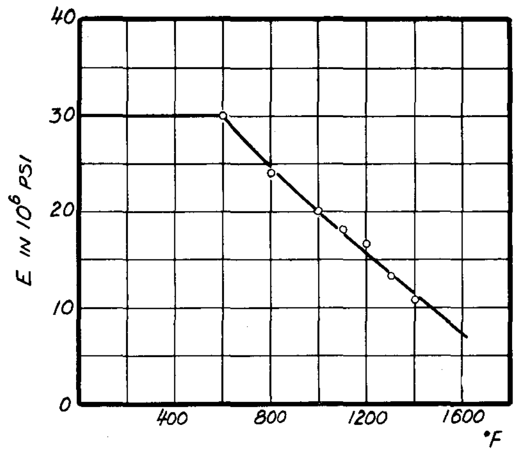

FIG. 2 YOUNG'S MODULUS AS A FUNCTION OF TEMPERATURE

using Noland's data. First, the position of the median surface will be determined, using Equation [10]. This is found to be 0.588 b from the inside surface. Next, by taking the cold cylinder to be at 100° F, the ratio of the thicknesses of the hot cylinder and the equivalent cold cylinder is computed by using Equations [6], [7], and [29]. It is found that

$$\bar{b} = 0.936b$$

Thus the equivalent cold cylinder of the same material is 93.6 per cent as thick as the hot cylinder.

The ratio of the loads on the cold cylinder and the hot cylinder is controlled by $\bar{D}_0/D_0$. This is computed as

$$\bar{D}_0/D_0 = 1.29$$

By taking $\alpha = 10^{-5}/°\mathrm{F}$, and $b = 0.095$ in., the values of thermal stresses are

$$N_T = 22400 \text{ lb/in.}, \qquad n_T = 0.958 \times 10^{-2}$$
$$M_T = 192 \text{ lb.} \qquad m_T = -0.124$$

If the radius of the cylinder is 2.25 in., and if the internal pressure P^0 is 1500 psi, then

$$p^0 = 1.44 \times 10^{-3}$$

If the axial tension $N_x{}^0$ is due to the same internal pressure, then

$$n_\xi{}^0 = \frac{p^0}{2} = 0.72 \times 10^{-3}$$

Now

$$\beta = 12\left(\frac{R}{b}\right)^2$$

Therefore the ratio of m_T/β and $n_\xi{}^0$ is

$$m_T/\beta n_\xi{}^0 = -0.0224$$

This is indeed smaller than the thickness-radius ratio $b/R = 0.0422$. Hence the surmise that m_T/β is negligible against $n_\xi{}^0$ is now justified by numerical calculation.

Junction Stress Between Cylinder and Head

The previous formulation of the similarity law for stresses in the hot cylinder and the cold cylinder is based upon the assumption that the secondary load system is specified. This is not true for the junction stresses induced by fitting, say, a hemispherical head to the cylinder. Such stresses are determined by the equality of deformations of the head and the cylinder at the junction. If the semi-spherical shell has the same thickness as the cylindrical shell, then the temperature distribution in the spherical shell will be the same as in the cylindrical shell. An analysis shows that the "similar" cold test specimen can be made also of uniform thickness $\bar{b}$, determined by Equation [29]. However, the similarity of both the primary loading and secondary loading now requires an additional restriction: ϵ must now be unity. When these conditions are fulfilled, the similarity of junction stresses in the hot cylinder and the cold cylinder will be assured, and the relations previously developed for computing the stresses in the hot cylinder from test data on the cold cylinder remain valid.

Ring Stiffener Around Cylinder

To strengthen the thin cylindrical shell against concentrated loads from the mounting lugs, a ring stiffener is often attached to the outside surface. It remains cold. Young's modulus of the ring material thus is that of cold material. To determine the dimensions of a "similar" ring for the cold cylinder, the conditions to be satisfied are those of Equations [31]. In other words, for a ratio ϵ of deformation of the ring on the cold cylinder to that on the hot cylinder, the force required must bear the ratio $\epsilon\bar{D}_0/D_0$. This means that the ratio of the stiffness of the ring for the cold cylinder to that on the hot cylinder is $\bar{D}_0/D_0$. If the ring is rectangular in section, the correct ratio of the width of the two rings for complete similarity is then $\bar{D}_0/D_0$. The ring on the cold cylinder is thus wider than on the hot cylinder. When this condition on the ring dimension is satisfied, the simple similarity relations for the stresses in the shell are again correct.

References

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